DESAIN DIDAKTIS KONSEP PECAHAN BERBASIS TEORI APOS UNTUK MENGEMBANGKAN ARGUMENTASI MATEMATIS SISWA SEKOLAH DASAR

DISERTASI

diajukan untuk memenuhi sebagian syarat untuk memperoleh gelar
Doktor Pendidikan Dasar

oleh

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PROGRAM STUDI
PENDIDIKAN DASAR
SEKOLAH PASCASARJANA
UNIVERSITAS PENDIDIKAN INDONESIA
2023

DESAIN DIDAKTIS KONSEP PECAHAN BERBASIS TEORI APOS UNTUK MENGEMBANGKAN ARGUMENTASI MATEMATIS SISWA SEKOLAH DASAR

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2023

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Dengan ini saya menyatakan bahwa disertasi dengan judul "Desain Didaktis Konsep Pecahan Berbasis Teori APOS untuk Mengembangkan Argumentasi Matematis Siswa Sekolah Dasar" ini beserta seluruh isinya adalah benar-benar karya saya sendiri. Saya tidak melakukan penjiplakan atau pengutipan dengan cara-cara yang tidak sesuai dengan etika ilmu yang berlaku dalam masyarakat keilmuan. Atas pernyataan ini, saya siap menanggung risiko/sanksi apabila di kemudian hari ditemukan adanya pelanggaran etika keilmuan atau ada klaim dari pihak lain terhadap keaslian karya ini.

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Desain Didaktis Konsep Pecahan Berbasis Teori APOS Untuk Mengembangkan Argumentasi Matematis Siswa Sekolah Dasar

ABSTRAK

Pecahan merupakan salah satu topik matematika yang kompleks dan sulit dipahami peserta didik. Pecahan bukanlah sebuah konsep tunggal tetapi memiliki lima konstruk pengertian yang saling berkaitan yaitu bagian dari keseluruhan, rasio, operator, hasil bagi, and ukuran. Penelitian ini bertujuan untuk mengembangkan desain didaktis melalui proses formulasi dan sintesis kelima konstruk pecahan sehingga dihasilkan pemahaman siswa yang mendalam. Teori yang digunakan sebagai basis penyusunan desain didaktis ini adalah Teori APOS (aksi, proses, objek, dan skema). Sementara argumentasi matematis digunakan untuk melihat penalaran siswa dan tolak ukur keberhasilan desain didaktis. Pengembangan desain didaktis dilakukan dengan metodologi penelitian kualitatif *Didactical Design Research* yang terdiri dari tiga tahap yaitu analisis prospektif, analisis metapedadidaktik, dan analisis retrospektif. Penelitian melibatkan 103 siswa dan 15 guru Kelas IV Sekolah Dasar yang tersebar di Kota Singaraja Provinsi Bali. Data dikumpulkan melalui tes, wawancara, observasi, dokumentasi, dan perangkat pembelajaran. Pada analisis prospektif menemukan bahwa siswa dan guru memiliki pengetahuan yang rendah dan bahkan miskonsepsi pada konsep dasar pecahan. Hasil analisis juga menunjukkan bahwa guru memiliki pengetahuan didaktis yang terbatas. Sintesis hasil analisis pengetahuan siswa dan guru dalam konsep pecahan menemukan bahwa siswa mengalami hambatan belajar epistemologis, didaktis, dan ontogenik. Berdasarkan hasil analisis hambatan belajar, selanjutnya disusun lintasan belajar hipotetik yang terdiri dari lima topik pembelajaran, masing-masing terdiri dari situasi didaktis dan lembar aktivitas siswa. Hasil analisis tahap metapedadidaktik dan restrospektif menunjukkan bahwa desain didaktis yang dikembangkan terbukti mampu mengembangkan argumentasi matematis siswa yang ditunjukkan dengan keberhasilan siswa menjawab soal dengan aspek klaim, data, bukti dan dukungan yang tepat.

Kata kunci: Desain Didaktis, Pecahan, Teori APOS, Argumentasi Matematis.

Didactic Design of Fraction Concept Based on APOS Theory to Develop Elementary Students' Mathematical Argumentation

ABSTRACT

Fractions are a complex and difficult topic for students to understand in mathematics. Fractions are not a single concept but consist of five interrelated constructs, namely part of the whole, ratio, operator, quotient, and measurement. This study aims to develop a didactic design through the formulation and synthesis of these five fraction constructs, with the goal of deepening students' understanding. The APOS theory (action, process, object, and schema) forms the theoretical basis for the development of this didactic design. Mathematical argumentation serves as an analytical tool to examine students' reasoning and gauge the efficacy of the didactic design. The development of the didactic design was carried out using the qualitative research methodology known as Didactical Design Research (DDR), which comprises three stages: prospective analysis, metapedadidactic analysis, and retrospective analysis. The study involved 103 students and 15 fourth-grade teachers from elementary schools in Singaraja, Bali. Data were collected through tests, interviews, observations, documentation, and learning tools. The prospective analysis revealed that both students and teachers had low knowledge and even misconceptions about the basic concepts of fractions. The results also demonstrated that teachers had limited didactical knowledge. The synthesis of the analysis of students' and teachers' knowledge about fractions identified epistemological, didactical, and ontogenetic learning obstacles. Based on the analysis of these obstacles, a hypothetical learning trajectory was developed, consisting of five learning topics. Each topic included didactic situations and student worksheets. The results from the metapedadidactic and retrospective analysis stages show that the developed didactic design succeeded in cultivating students' mathematical argumentation, as demonstrated by the students' success in answering questions with appropriate claims, data, warrants, and backing.

Kata kunci: Didactic Design, Fractions, APOS Theory, Mathematical Argumentation.

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