BAB III

DEFICIENCY

Pembahasaan *deficiency* merupakan bagian dari pembahasan teori sampel besar. *Deficiency* sendiri adalah suatu metode untuk membandingkan dua buah penaksir pada ukuran sampel besar dilihat dari nilai MSE penaksir tersebut. Maka yang pertama kali harus dicari untuk menentukan *deficiency* adalah menentukan nilai dari MSE kedua penaksir.

3.1 Asumsi-Asumsi dan Hasil-Hasil Perhitungan

Sebelum membahas lebih lanjut mengenai d*eficiency*, akan diperkenalkan terlebih dahulu beberapa asumsi dan hasil-hasil (tanpa bukti) yang akan digunakan untuk menjelaskan mengenai teori *deficiency*.

Diketahui fungsi kepadatan peluang dari distribusi keluarga eksponensial adalah

$$f\left(x;\theta\right)=\exp\left\{\phi_1\left(\theta\right)T\left(x\right)+\phi_2\left(\theta\right)+Q\left(x\right)\right\}; \qquad x\in\Re,\theta\in\Theta \tag{3.1}$$

3.1.1 Asumsi

Misalkan variabel acak $X_1, X_2,\ldots,X_n$ iid pada (3.1). Dan misalkan $g\left(\theta\right)$ adalah fungsi yang estimable untuk θ. Maka berlaku asumsi berikut:

a. Persamaan (3.1) memenuhi kondisi

$$\eta\left(\theta\right)\phi_1'\left(\theta\right)+\phi_2'\left(\theta\right)=0 \text{ dan } \phi_1'\left(\theta\right)>0, \quad \forall\theta\in\Theta.$$

dimana $\eta(\theta)$ adalah suatu fungsi dari θ.

b. θ^* adalah statistik cukup untuk distribusi keluarga eksponensial, dimana

$\theta^* = \left(\frac{1}{n}\right)\sum_{i=1}^{n} T(x_i)$ dan $E(\theta^*) = \eta(\theta)$

c. Fungsi log-likelihood $l(\theta) = \log\left[L_n(\underline{x},\theta)\right]$ adalah unimodal dan penaksir maksimum Likelihood yang merupakan fungsi dari θ^* adalah unik. Sehingga $g(\theta^*)$ adalah penaksir Maksimum Likelihood dari $g(\theta)$ (Zehna, 1966).

d. $U(\theta^*)$ adalah penaksir UMVU dari $g(\theta)$, dimana penaksir UMVU adalah fungsi dari θ^*.

e. Penaksir Maksimum Likelihood $g(\theta^*)$ dan penaksir UMVU $U(\theta^*)$ dapat menjadi identik, dengan kata lain $g(\theta^*) = U(\theta^*)$, jika parameter natural dari distribusi keluarga eksponensial adalah θ [Greenwood and Nikulin, 1996]. Secara umum ML dan UMVU berbeda.

f. Fungsi $g(\theta^*)$ dan $U(\theta^*)$ konvergen pada ekspansi taylor, untuk semua titik dalam Θ.

g. Diasumsikan terdapat turunan dari $g(\theta^*)$ dan $U(\theta^*)$ pada ekspansi taylor

h. Diasumsikan penaksir Maksimum likelihood *asymptotically efficient*, yaitu mencapai batas bawah dari Cramer-Rao ketika ukuran sampel besar dan menuju tak hingga. Hal ini berarti tidak ada penaksir tak bias yang memiliki nilai MSE lebih kecil dibanding penaksir Maksimum Likelihood.

i. Untuk setiap penaksir yang *asymptotically efficient,* berlaku

$$\sqrt{n}\left(\theta^{*}-\theta\right)\xrightarrow{\ell} N\left(0,\frac{1}{I(\theta)}\right)$$

j. Jika I adalah informasi Fisher seperti pada definisi 2.16, maka menurut (Gudi dan Nagnur, 2004) $\frac{1}{I}$ memiliki turunan terhadap θ, yaitu $(d/d\theta)[1/I]=-(2K_{11}+K_{30})/I^{2}$

3.1.2 Hasil-Hasil Perhitungan (Tanpa Bukti)

Misalkan $l'(\theta)$ adalah turunan pertama dari $l(\theta)$ terhadap θ. Seperti diketahui jika pada persamaan likelihood $l'(\theta)=0$, maka penaksiran tersebut memiliki solusi θ^{*}, yang memiliki peluang mendekati θ. Hal ini juga membuat fungsi likelihood maksimal.

Dari asumsi (3.1.1.f), (3.1.1.g) dan (3.1.1.h) dan menggunakan hasil dari (Gudi dan Nagnur, 2004) maka diperoleh

$$\sqrt{n}\left(\theta^{*}-\theta\right)=\frac{\{l'(\theta)\}}{\sqrt{nI}}+\frac{\{l'(\theta)\}\{l''(\theta)+nI\}}{n^{3/2}I^{2}}+\frac{\{l'(\theta)\}^{2}\{l'''(\theta)\}}{2n^{5/2}I^{3}}+O\left(n^{-5/2}\right)$$

$$=\frac{\{l'(\theta)\}}{\sqrt{nI}}+\frac{\{l'(\theta)\}\{l''(\theta)+nI\}}{n^{3/2}I^{2}}+\frac{\{l'(\theta)\}^{2}E\{l'''(\theta)\}}{2n^{5/2}I^{3}}$$

$$+\frac{\{l'(\theta)\}^2\left[\{l'''(\theta)\}-E\{l'''(\theta)\}\right]}{2n^{5/2}I^3}+O\left(n^{-5/2}\right) \qquad (3.2)$$

Misalkan

$$K_{ij}=E\left[\left\{\frac{\partial \log f(X,\theta)}{\partial\theta}\right\}^i\left\{\frac{\partial^2 \log f(X,\theta)}{\partial\theta}+I\right\}^j\right] \qquad (3.3)$$

Maka,

$$E\left[\{l'(\theta)\}^i\{l''(\theta)+nI\}^j\right]=nK_{ij} \qquad (3.4)$$

Dengan menggunakan hasil pada (Cox dan Hinkley, 1974)

$$\begin{aligned}E\left[\{l'''(\theta)\}\right] &= -3E\left[\{l'(\theta)\}\{l''(\theta)+nI\}\right]-E\left[\{l'(\theta)\}\right]^3\\ &= -n\left(3K_{11}+K_{30}\right)\end{aligned}$$

Maka persamaan (3.2) dapat dirubah menjadi

$$\sqrt{n}\left(\theta^*-\theta\right)=\frac{\{l'(\theta)\}}{\sqrt{nI}}+\frac{\{l'(\theta)\}\{l''(\theta)+nI\}}{n^{3/2}I^2}-\frac{\{l'(\theta)\}^2\left(3K_{11}+K_{11}\right)}{2n^{3/2}I^3}+O\left(n^{-5/2}\right) \qquad (3.5)$$

Dari (3.5) dapat dicari pendekatan moment dari $\sqrt{n}\left(\theta^*-\theta\right)$ pada order ke n^{-2}.

Misalkan $\mu_i=E\left[\theta^*-\theta\right]^i$; *untuk i= 1, 2, 3, 4*. Dengan menggunakan hasil pada (Aithal, 1992; Gudi, 2002; Rao, 1961) diperoleh

1. $\mu_0 \quad = 1$ (3.6)
2. $\mu_1 \quad = E\left(\theta^*-\theta\right)$

$$= \frac{b(\theta)}{n} = -\frac{(K_{11} + K_{30})}{2nI^2} + O(n^{-2}) \tag{3.7}$$

3. $\mu_2 \quad = E(\theta^* - \theta)^2$

$$= \frac{1}{nI} + \frac{2b'(\theta)}{n^2 I} + \frac{\psi(\theta)}{n^2} + \frac{[b(\theta)]^2}{n^2} + O(n^{-3}) \tag{3.8}$$

4. $\mu_3 \quad = E(\theta^* - \theta)^3$

$$= -\frac{(9K_{11} + 7K_{30})}{2n^2 I^3} + O(n^{-3}) \tag{3.9}$$

5. $\mu_4 \quad = E(\theta^* - \theta)^4$

$$= \frac{3}{n^2 I^2} + O(n^{-3}) \tag{3.10}$$

Dimana $\frac{b(\theta)}{n}$ adalah order bias pertama dari penaksir θ^*, $b'(\theta)$ adalah turunan dari $b(\theta)$ terhadap θ, $\psi(\theta)$ adalah koefisien dari n^{-2} pada varians dari penaksir $\hat{\theta}$ (yaitu, penaksir θ^* dikoreksi untuk bias order yang pertama) dan ditunjukan dengan

$$\psi(\theta) = \left[\frac{2\left[IK_{02} - K_{11}^2\right] + \left[K_{11} + K_{30}\right]^2}{2I^4} \right]$$

Sehingga,

$$\operatorname{var}\left(\theta^*\right) = E\left(\theta^* - \theta\right)^2 - \left(E\left(\theta^* - \theta\right)\right)^2 \tag{3.11}$$

Dengan mensubtitusi persamaan (3.7) dan (3.8) pada persamaan (3.11) diperoleh

$$\begin{aligned} \operatorname{var}\left(\theta^*\right) &= \left(\frac{1}{nI} + \frac{2b'(\theta)}{n^2 I} + \frac{\psi(\theta)}{n^2} + \frac{\left[b(\theta)\right]^2}{n^2} + O\left(n^{-3}\right)\right) - \left(\frac{b(\theta)}{n} + O\left(n^{-2}\right)\right)^2 \\ &= \frac{1}{nI} + \frac{2b'(\theta)}{n^2 I} + \frac{\psi(\theta)}{n^2} + O\left(n^{-3}\right) \end{aligned} \tag{3.12}$$

Dari definisi MSE diketahui

$$MSE\left(\theta^*\right) = \operatorname{var}\left(\theta^*\right) + \left(Bias\left(\theta^*, \theta\right)\right)^2 \tag{3.13}$$

Dengan mensubtitusi (3.7) dan (3.12) pada (3.13) diperoleh

$$MSE\left(\theta^*\right) = \frac{1}{nI} + \frac{2b'(\theta)}{n^2 I} + \frac{\psi(\theta)}{n^2} + \frac{\left(b(\theta)\right)^2}{n^2} + O\left(n^{-3}\right) \tag{3.14}$$

3.2 Perhitungan *Means Square Error* untuk Penaksir Maximum Likelihood dan Penaksir UMVU

Seperti telah disebutkan sebelumnya, *deficiency* ditentukan dari nilai MSE dari kedua penaksir. Maka langkah berikut adalah menentukan nilai MSE dari kedua buah penaksir.

3.2.1 Perhitungan *Means Square Error* untuk Penaksir Maximum Likelihood

Dengan menggunakan asumsi bahwa terdapat turunan dari $g\left(\theta^{*}\right)$ dan $U\left(\theta^{*}\right)$ pada ekspansi taylor, maka dapat diperlihatkan rangkaian Ekspansi Taylor dari $g\left(\theta^{*}\right)$ diberikan oleh

$$g\left(\theta^{*}\right) = g(\theta) + \frac{g'(\theta)\left(\theta^{*}-\theta\right)}{1!} + \frac{g''(\theta)\left(\theta^{*}-\theta\right)^{2}}{2!} + \frac{g'''(\theta)\left(\theta^{*}-\theta\right)^{3}}{3!} + \frac{g''''(\theta)\left(\theta^{*}-\theta\right)^{4}}{4!} + \cdots \quad (3.15)$$

Dimana $g^{(i)}(\theta), i = 1, 2, \ldots$; adalah turunan ke-*i* dari $g(\theta)$ terhadap θ.

Order bias yang pertama, varians, dan MSE dari penaksir Maksimum likelihood $g\left(\theta^{*}\right)$ dapat dihitung dengan menggunakan persamaan (3.15)

Lemma 3.2.1

Order bias yang pertama dari penaksir $g\left(\theta^{*}\right)$ adalah

$$E\left[g\left(\theta^{*}\right) - g(\theta)\right] = \left\{g'(\theta)\right\}\left[-\frac{\left(K_{11}-K_{30}\right)}{2nI^{2}}\right] + \left\{g''(\theta)\right\}\left[\frac{1}{2nI}\right] \quad (3.16)$$

Bukti

Dengan mengambil ekspektasi pada kedua sisi pada persamaan (3.15) diperoleh

$$E\left[g\left(\theta^{*}\right)\right] = E\left[g(\theta) + \frac{g'(\theta)\left(\theta^{*}-\theta\right)}{1!} + \frac{g''(\theta)\left(\theta^{*}-\theta\right)^{2}}{2!} + \frac{g'''(\theta)\left(\theta^{*}-\theta\right)^{3}}{3!}\right.$$

$$+\frac{g''''(\theta)\left(\theta^*-\theta\right)^4}{4!}+\cdots\Bigg]$$

$$E\left[g\left(\theta^*\right)\right]=g(\theta)+\frac{g'(\theta)E\left[\left(\theta^*-\theta\right)\right]}{1!}+\frac{g''(\theta)E\left[\left(\theta^*-\theta\right)^2\right]}{2!}+\frac{g'''(\theta)E\left[\left(\theta^*-\theta\right)^3\right]}{3!}$$

$$+\frac{g''''(\theta)E\left[\left(\theta^*-\theta\right)^4\right]}{4!}+\cdots$$

Dengan mensubtitusi (3.7)-(3.10) pada persamaan terakhir, diperoleh

$$E\left[g\left(\theta^*\right)\right]=g(\theta)+\frac{g'(\theta)}{1}\mu_1+\frac{g''(\theta)}{2}\mu_2+\frac{\{g'''(\theta)\}}{6}\mu_1+\frac{\{g''''(\theta)\}}{24}\mu 4+\ldots$$

$$E\left[g\left(\theta^*\right)\right]=g(\theta)+g'(\theta)\left[-\frac{(K_{11}+K_{30})}{2nI^2}\right]+\frac{g''(\theta)}{2}\left[\frac{1}{nI}+\frac{2b'(\theta)}{n^2I}+\frac{\psi(\theta)}{n^2}+\frac{\left[b(\theta)\right]^2}{n^2}\right]$$

$$+\frac{g'''(\theta)}{6}\left[-\frac{(9K_{11}+7K_{30})}{2n^2I^3}\right]+\frac{g''''(\theta)}{24}\left[\frac{3}{n^2I^2}\right]+O\left(n^{-3}\right)$$

Dengan mengabaikan suku dengan order kurang dari $O\left(n^{-1}\right)$, diperoleh

$$E\left[g\left(\theta^*\right)\right]=g(\theta)-g'(\theta)\left[\frac{(K_{11}+K_{30})}{2nI^2}\right]+\frac{g''(\theta)}{2}\left[\frac{1}{nI}\right]+O\left(n^{-2}\right) \qquad (3.17)$$

Sehingga, order bias pertama dari $g\left(\theta^*\right)$ adalah,

$$E\left[g\left(\theta^*\right)-g(\theta)\right]=\frac{b\left[g(\theta)\right]}{n}$$

$$=-g'(\theta)\left[\frac{(K_{11}+K_{30})}{2nI^2}\right]+\frac{g''(\theta)}{2}\left[\frac{1}{nI}\right]+O\left(n^{-2}\right)$$

$$= g'(\theta)\left[-\frac{(K_{11}+K_{30})}{2nI^2}\right]+\{g''(\theta)\}\left[\frac{1}{2nI}\right] \qquad (3.18)$$

Untuk selanjutnya $E\left[g\left(\theta^*\right)-g\left(\theta\right)\right]$ ditulis $\frac{b\left[g(\theta)\right]}{n}$.

Teorema 1

Varians dari penaksir Maksimum likelihood $g\left(\theta^*\right)$ adalah

$$\text{var}\left[g\left(\theta^*\right)\right]=\{g'(\theta)\}^2\left[\text{var}\left(\theta^*\right)\right]-\{g'(\theta)\}\{g''(\theta)\}\left[\frac{(4K_{11}+3K_{30})}{n^2I^3}\right]+\{g''(\theta)\}^2\left[\frac{1}{2n^2I^2}\right]$$

$$+\{g'(\theta)\}\{g'''(\theta)\}\left[\frac{1}{n^2I^2}\right]+O\left(n^{-3}\right) \qquad (3.19)$$

Bukti

Dari definisi varians diketahui bahwa

$$\text{var}\left[g\left(\theta^*\right)\right]=E\left[g\left\{\theta^*\right\}-E\left\{g\left(\theta^*\right)\right\}\right]^2$$

Dengan mengurangi persamaan (3.15) oleh (3.17) diperoleh

$$\left[\left\{g\left(\theta^*\right)\right\}-E\left\{g\left(\theta^*\right)\right\}\right]$$

$$=\left[g(\theta)+\frac{g'(\theta)\left(\theta^*-\theta\right)}{1!}+\frac{g''(\theta)\left(\theta^*-\theta\right)^2}{2!}+\frac{g'''(\theta)\left(\theta^*-\theta\right)^3}{3!}+\frac{g''''(\theta)\left(\theta^*-\theta\right)^4}{4!}+\cdots\right]-$$

$$\left[g(\theta)-g'(\theta)\left[\frac{(K_{11}+K_{30})}{2nI^2}\right]+\frac{g''(\theta)}{2}\left[\frac{1}{nI}\right]+O\left(n^{-2}\right)\right]$$

$$= \left[\frac{g'(\theta)(\theta^* - \theta)}{1!} + \frac{g''(\theta)(\theta^* - \theta)^2}{2} + \frac{g'''(\theta)(\theta^* - \theta)^3}{6} + \frac{g''''(\theta)(\theta^* - \theta)^4}{24} + \cdots \right] +$$

$$g'(\theta)\left[\frac{K_{11} + K_{30}}{2nI^2}\right] - \{g''(\theta)\}\left[\frac{1}{2nI}\right] + O\left(n^{-2}\right) \qquad (3.20)$$

Kuadratkan persamaan (3.20) di atas, diperoleh

$$\left[\left\{g\left(\theta^*\right)\right\} - E\left\{g\left(\theta^*\right)\right\}\right]^2$$

$$= \left[g'(\theta)^2 \left(\theta - \theta^*\right)^2\right] + \left[g'(\theta)\frac{g''(\theta)}{2}\left(\theta - \theta^*\right)\left(\theta - \theta^*\right)^2\right] + \left[g'(\theta)\frac{g''(\theta)}{6}\left(\theta - \theta^*\right)\left(\theta - \theta^*\right)^3\right]$$

$$+ \left[g'(\theta)\frac{g''''(\theta)}{24}\left(\theta - \theta^*\right)\left(\theta - \theta^*\right)^4\right] + \ldots + \left[g'(\theta)^2\left(\theta - \theta^*\right)\left[\frac{K_{11} + K_{30}}{2nI^2}\right]\right] + \left[g'(\theta)\right.$$

$$\left.\{-g''(\theta)\}\left(\theta - \theta^*\right)\left[\frac{1}{2nI}\right]\right] + \left[\frac{g''(\theta)}{2} g'(\theta)\left(\theta - \theta^*\right)^2\left(\theta - \theta^*\right)\right] + \left[\left(\frac{g''(\theta)}{2}\right)^2\left(\left(\theta - \theta^*\right)^2\right)^2\right]$$

$$+ \left[\frac{g''(\theta)}{2}\frac{g'''(\theta)}{6}\left(\theta - \theta^*\right)^2\left(\theta - \theta^*\right)^3\right] + \left[\frac{g''(\theta)}{2}\frac{g''''(\theta)}{24}\left(\theta - \theta^*\right)^2\left(\theta - \theta^*\right)^4\right] + \ldots$$

$$+ \left[\frac{g''(\theta)}{2} g'(\theta)\left(\theta - \theta^*\right)^2\left[\frac{K_{11} + K_{30}}{2nI^2}\right]\right] + \left[\frac{g''(\theta)}{2}\{-g''(\theta)\}\left(\theta - \theta^*\right)^2\left[\frac{1}{2nI}\right]\right]$$

$$+ \left[\frac{g'''(\theta)}{6} g'(\theta)\left(\theta - \theta^*\right)^3\left(\theta - \theta^*\right)\right] + \left[\frac{g'''(\theta)}{6}\frac{g''(\theta)}{2}\left(\theta - \theta^*\right)^3\left(\theta - \theta^*\right)^2\right] +$$

$$+ \left[\left(\frac{g'''(\theta)}{6}\right)^2\left(\left(\theta - \theta^*\right)^3\right)^2\right] + \left[\frac{g'''(\theta)}{6}\frac{g''''(\theta)}{24}\left(\theta - \theta^*\right)^3\left(\theta - \theta^*\right)^4\right] + \ldots$$

$$+\left[\frac{g'''(\theta)}{6}g'(\theta)\left(\theta-\theta^*\right)^3\left[\frac{K_{11}+K_{30}}{2nI^2}\right]\right]+\left[\frac{g'''(\theta)}{6}\left\{-g''(\theta)\right\}\left(\theta-\theta^*\right)^3\left[\frac{1}{2nI}\right]\right]$$

$$+\left[\frac{g''''(\theta)}{24}g'(\theta)\left(\theta-\theta^*\right)^4\left(\theta-\theta^*\right)\right]+\left[\frac{g''''(\theta)}{24}\frac{g''(\theta)}{2}\left(\theta-\theta^*\right)^4\left(\theta-\theta^*\right)^2\right]+$$

$$+\left[\frac{g''''(\theta)}{24}\frac{g'''(\theta)}{6}\left(\theta-\theta^*\right)^4\left(\theta-\theta^*\right)^3\right]+\left[\left(\frac{g''''(\theta)}{24}\right)\left(\left(\theta-\theta^*\right)^4\right)^2\right]+\ldots$$

$$+\left[\frac{g''''(\theta)}{24}g'(\theta)\left(\theta-\theta^*\right)^4\left[\frac{K_{11}+K_{30}}{2nI^2}\right]\right]+\left[\frac{g''''(\theta)}{24}\left\{-g''(\theta)\right\}\left(\theta-\theta^*\right)^4\left[\frac{1}{2nI}\right]\right]$$

$$+\left[\left\{g'(\theta)\right\}^2\left(\theta-\theta^*\right)\left[\frac{K_{11}+K_{30}}{2nI^2}\right]\right]+\left[g'(\theta)\frac{g''(\theta)}{2}\left(\theta-\theta^*\right)^2\left[\frac{K_{11}+K_{30}}{2nI^2}\right]\right]+\left[g'(\theta)\frac{g''(\theta)}{2}\right.$$

$$\left.\left(\theta-\theta^*\right)^3\left[\frac{K_{11}+K_{30}}{2nI^2}\right]\right]+\left[g'(\theta)\frac{g''''(\theta)}{24}\left(\theta-\theta^*\right)^4\left[\frac{K_{11}+K_{30}}{2nI^2}\right]\right]+\ldots+\left[\left(g'(\theta)\right)^2\left[\frac{K_{11}+K_{30}}{2nI^2}\right]^2\right]$$

$$+\left[g'(\theta)\left\{-g''(\theta)\right\}\left[\frac{K_{11}+K_{30}}{2nI^2}\right]\left[\frac{1}{2nI}\right]\right]+\left[\left(-g''(\theta)\right)g'(\theta)\left[\frac{1}{2nI}\right]\left(\theta-\theta^*\right)\right]$$

$$+\left[\left(-g''(\theta)\right)\frac{g''(\theta)}{2}\left(\theta-\theta^*\right)^2\left[\frac{1}{2nI}\right]\right]+\left[\left(-g''(\theta)\right)\frac{g'''(\theta)}{6}\left(\theta-\theta^*\right)^3\left[\frac{1}{2nI}\right]\right]$$

$$+\left[\left(-g''(\theta)\right)\frac{g''''(\theta)}{24}\left(\theta-\theta^*\right)^4\left[\frac{1}{2nI}\right]\right]+\ldots+\left[\left(-g''(\theta)\right)g'(\theta)\left[\frac{1}{2nI}\right]\left[\frac{K_{11}+K_{30}}{2nI^2}\right]\right]$$

$$+\left[\left(-g''(\theta)\right)^2\left[\frac{1}{2nI}\right]^2\right]+O\left(n^{-4}\right) \qquad (3.21)$$

Dengan mengambil ekspektasi pada kedua sisi persamaan (3.21) diperoleh

$$\operatorname{var}\left[g\left(\theta^*\right)\right]= E\left[\left\{g\left(\theta^*\right)\right\}-E\left\{g\left(\theta^*\right)\right\}\right]^2$$

$$= \left[g'(\theta)^2 E\left(\theta-\theta^*\right)^2\right]+\left[g'(\theta)\frac{g''(\theta)}{2}E\left(\theta-\theta^*\right)E\left(\theta-\theta^*\right)^2\right]+\left[g'(\theta)\frac{g''(\theta)}{6}E\left(\theta-\theta^*\right)\right.$$

$$\left.E\left(\theta-\theta^*\right)^3\right]$$

$$+\left[g'(\theta)\frac{g''''(\theta)}{24}E\left(\theta-\theta^*\right)E\left(\theta-\theta^*\right)^4\right]+\ldots+\left[g'(\theta)^2 E\left(\theta-\theta^*\right)\left[\frac{K_{11}+K_{30}}{2nI^2}\right]\right]+\left[g'(\theta)\right.$$

$$\left.\left\{-g''(\theta)\right\}E\left(\theta-\theta^*\right)\left[\frac{1}{2nI}\right]\right]+\left[\frac{g''(\theta)}{2}g'(\theta)E\left(\theta-\theta^*\right)^2 E\left(\theta-\theta^*\right)\right]+\left[\left(\frac{g''(\theta)}{2}\right)^2 E\left(\left(\theta-\theta^*\right)^2\right)^2\right]$$

$$+\left[\frac{g''(\theta)}{2}\frac{g'''(\theta)}{6}E\left(\theta-\theta^*\right)^2 E\left(\theta-\theta^*\right)^3\right]+\left[\frac{g''(\theta)}{2}\frac{g''''(\theta)}{24}E\left(\theta-\theta^*\right)^2 E\left(\theta-\theta^*\right)^4\right]+\ldots$$

$$+\left[\frac{g''(\theta)}{2}g'(\theta)E\left(\theta-\theta^*\right)^2\left[\frac{K_{11}+K_{30}}{2nI^2}\right]\right]+\left[\frac{g''(\theta)}{2}\left\{-g''(\theta)\right\}E\left(\theta-\theta^*\right)^2\left[\frac{1}{2nI}\right]\right]$$

$$+\left[\frac{g'''(\theta)}{6}g'(\theta)E\left(\theta-\theta^*\right)^3 E\left(\theta-\theta^*\right)\right]+\left[\frac{g'''(\theta)}{6}\frac{g''(\theta)}{2}E\left(\theta-\theta^*\right)^3 E\left(\theta-\theta^*\right)^2\right]+$$

$$+\left[\left(\frac{g'''(\theta)}{6}\right)^2 E\left(\left(\theta-\theta^*\right)^3\right)^2\right]+\left[\frac{g'''(\theta)}{6}\frac{g''''(\theta)}{24}E\left(\theta-\theta^*\right)^3 E\left(\theta-\theta^*\right)^4\right]+\ldots$$

$$+\left[\frac{g'''(\theta)}{6}g'(\theta)E\left(\theta-\theta^*\right)^3\left[\frac{K_{11}+K_{30}}{2nI^2}\right]\right]+\left[\frac{g'''(\theta)}{6}\left\{-g''(\theta)\right\}E\left(\theta-\theta^*\right)^3\left[\frac{1}{2nI}\right]\right]$$

$$+\left[\frac{g''''(\theta)}{24}g'(\theta)E\left(\theta-\theta^*\right)^4 E\left(\theta-\theta^*\right)\right]+\left[\frac{g''''(\theta)}{24}\frac{g''(\theta)}{2}E\left(\theta-\theta^*\right)^4 E\left(\theta-\theta^*\right)^2\right]+$$

$$+\left[\frac{g''''(\theta)}{24}\frac{g'''(\theta)}{6}E\left(\theta-\theta^*\right)^4 E\left(\theta-\theta^*\right)^3\right]+\left[\left(\frac{g''''(\theta)}{24}\right)E\left(\left(\theta-\theta^*\right)^4\right)^2\right]+\ldots$$

$$+\left[\frac{g''''(\theta)}{24}g'(\theta)E\left(\theta-\theta^*\right)^4\left[\frac{K_{11}+K_{30}}{2nI^2}\right]\right]+\left[\frac{g''''(\theta)}{24}\left\{-g''(\theta)\right\}E\left(\theta-\theta^*\right)^4\left[\frac{1}{2nI}\right]\right]$$

$$+\left[\left\{g'(\theta)\right\}^2 E\left(\theta-\theta^*\right)\left[\frac{K_{11}+K_{30}}{2nI^2}\right]\right]+\left[g'(\theta)\frac{g''(\theta)}{2}E\left(\theta-\theta^*\right)^2\left[\frac{K_{11}+K_{30}}{2nI^2}\right]\right]$$

$$+\left[g'(\theta)\frac{g''(\theta)}{2}E\left(\theta-\theta^*\right)^3\left[\frac{K_{11}+K_{30}}{2nI^2}\right]\right]+\left[g'(\theta)\frac{g''''(\theta)}{24}E\left(\theta-\theta^*\right)^4\left[\frac{K_{11}+K_{30}}{2nI^2}\right]\right]+\ldots$$

$$+\left[\left(g'(\theta)\right)^2\left[\frac{K_{11}+K_{30}}{2nI^2}\right]^2\right]$$

$$+\left[g'(\theta)\left\{-g''(\theta)\right\}\left[\frac{K_{11}+K_{30}}{2nI^2}\right]\left[\frac{1}{2nI}\right]\right]+\left[\left(-g''(\theta)\right)g'(\theta)\left[\frac{1}{2nI}\right]E\left(\theta-\theta^*\right)\right]$$

$$+\left[\left(-g''(\theta)\right)\frac{g''(\theta)}{2}E\left(\theta-\theta^*\right)^2\left[\frac{1}{2nI}\right]\right]+\left[\left(-g''(\theta)\right)\frac{g'''(\theta)}{6}E\left(\theta-\theta^*\right)^3\left[\frac{1}{2nI}\right]\right]$$

$$+\left[\left(-g''(\theta)\right)\frac{g''''(\theta)}{24}E\left(\theta-\theta^*\right)^4\left[\frac{1}{2nI}\right]\right]+\ldots+\left[\left(-g''(\theta)\right)g'(\theta)\left[\frac{1}{2nI}\right]\left[\frac{K_{11}+K_{30}}{2nI^2}\right]\right]$$

$$+\left[\left(-g''(\theta)\right)^2\left[\frac{1}{2nI}\right]^2\right]+O\left(n^{-3}\right) \qquad (3.22)$$

Setelah dilakukan penyederhanaan dan mensubtitusi nilai-nilai μ_i, *i=1, 2, 3, 4* , dari hasil sebelumnya, maka persamaan (3.22) menjadi

$$\operatorname{var}\left[g\left(\theta^*\right)\right]=\left\{g'(\theta)\right\}^2\left[\frac{1}{nI}+\frac{2b'(\theta)}{n^2I}+\frac{\psi(\theta)}{n^2}\right]-\left\{g'(\theta)\right\}\left\{g''(\theta)\right\}\left[\frac{\left(4K_{11}+3K_{30}\right)}{n^2I^3}\right]$$

$$+\left\{g''(\theta)\right\}^2\left[\frac{1}{2n^2I^2}\right]+\left\{g'(\theta)\right\}\left\{g'''(\theta)\right\}\left[\frac{1}{n^2I^2}\right]+O\left(n^{-3}\right) \qquad \blacksquare$$

Sehingga dapat di simpulkan bahwa,

$$\operatorname{var}\left[g\left(\theta^*\right)\right]=\left\{g'(\theta)\right\}^2\left[\frac{1}{nI}+\frac{2b'(\theta)}{n^2I}+\frac{\psi(\theta)}{n^2}\right]-\left\{g'(\theta)\right\}\left\{g''(\theta)\right\}\left[\frac{\left(4K_{11}+3K_{30}\right)}{n^2I^3}\right]$$

$$+\left\{g''(\theta)\right\}^2\left[\frac{1}{2n^2I^2}\right]+\left\{g'(\theta)\right\}\left\{g'''(\theta)\right\}\left[\frac{1}{n^2I^2}\right]+O\left(n^{-3}\right) \qquad (3.23)$$

Dari definisi MSE, maka

$$MSE\left[g\left(\theta^*\right)\right] = \text{var}\left[g\left(\theta^*\right)\right] + \left(\frac{b\left[g\left(\theta\right)\right]}{n}\right)^2$$

$$= \Bigg[\{g'(\theta)\}^2\left[\frac{1}{nI}+\frac{2b'(\theta)}{n^2 I}+\frac{\psi(\theta)}{n^2}\right]-\{g'(\theta)\}\{g''(\theta)\}\left[\frac{(4K_{11}+3K_{30})}{n^2 I^3}\right]$$

$$+\{g''(\theta)\}^2\left[\frac{1}{2n^2 I^2}\right]+\{g'(\theta)\}\{g'''(\theta)\}\left[\frac{1}{n^2 I^2}\right]+O\left(n^{-3}\right)\Bigg]$$

$$+\left[g'(\theta)\left[-\frac{(K_{11}+K_{30})}{2nI^2}\right]+\{g''(\theta)\}\left[\frac{1}{2nI}\right]\right]^2$$

$$= \Bigg[\{g'(\theta)\}^2\left[\frac{1}{nI}+\frac{2b'(\theta)}{n^2 I}+\frac{\psi(\theta)}{n^2}\right]-\{g'(\theta)\}\{g''(\theta)\}\left[\frac{(4K_{11}+3K_{30})}{n^2 I^3}\right]$$

$$+\{g''(\theta)\}^2\left[\frac{1}{2n^2 I^2}\right]+\{g'(\theta)\}\{g'''(\theta)\}\left[\frac{1}{n^2 I^2}\right]+O\left(n^{-3}\right)\Bigg]$$

$$+\Bigg[g'(\theta)^2\left[-\frac{(K_{11}+K_{30})}{2nI^2}\right]^2+\{g''(\theta)\}^2\left[\frac{1}{2nI}\right]^2+2g'(\theta)\,g''(\theta)$$

$$\left[-\frac{(K_{11}+K_{30})}{2nI^2}\right]\left[\frac{1}{2nI}\right]\Bigg]$$

$$MSE\left[g\left(\theta^*\right)\right] = \{g'(\theta)\}^2\left[\frac{1}{nI}+\frac{2b'(\theta)}{n^2 I}+\frac{\psi(\theta)}{n^2}+\frac{(K_{11}+K_{30})^2}{4n^2 I^2}\right]-\left[\left(g'(\theta)\,g''(\theta)\right)\right]$$

$$\left[\frac{4K_{11}+3K_{30}}{n^2 I^3}+\frac{K_{11}+K_{30}}{2n^2 I^3}\right]+\left(g''(\theta)\right)^2\left[\frac{1}{2n^2 I^2}+\frac{1}{4n^2 I^2}\right]$$

$$+g'(\theta)\,g'''(\theta)\left[\frac{1}{n^2 I^2}\right] \tag{3.24}$$

3.2.2 Perhitungan *Means Square Error* untuk Penaksir UMVU

Misalkan $U\left(\theta^*\right)$ adalah penaksir UMVU dari $g(\theta)$, dengan asumsi $U\left(\theta^*\right)$ konvergen terhadap ekspansi Taylor, maka

$$U\left(\theta^*\right) = U(\theta) + \frac{U'(\theta)\left(\theta^*-\theta\right)}{1!} + \frac{U''(\theta)\left(\theta^*-\theta\right)^2}{2!} + \frac{U'''(\theta)\left(\theta^*-\theta\right)^3}{3!} + \frac{U''''(\theta)\left(\theta^*-\theta\right)^4}{4!} + \cdots \quad (3.25)$$

Dimana $U^{(i)}(\theta), i=1,2,\ldots;$ adalah turunan ke-*i* dari $U(\theta)$ terhadap θ. Perhitungan MSE dari penaksir UMVU $U\left(\theta^*\right)$ dapat dihitung dengan menggunakan persamaan (3.25).

Teorema 2

Mean Square Error (MSE) dari penaksir UMVU $U\left(\theta^*\right)$ adalah

$$\begin{aligned} MSE\left[U\left(\theta^*\right)\right] &= \operatorname{var}\left[U\left(\theta^*\right)\right] = E\left[U\left(\theta^*\right) - \left\{g(\theta)\right\}\right]^2 \\ &= \left\{g'(\theta)\right\}^2\left[\frac{1}{nI} + \frac{\psi(\theta)}{n^2}\right] + 2\left\{g'(\theta)\right\}\left\{g''(\theta)\right\}\left[-\frac{\left(K_{11}+K_{30}\right)}{n^2I^3}\right] \\ &\quad + \left\{g''(\theta)\right\}^2\left[\frac{1}{2n^2I^2}\right] + O\left(n^{-3}\right) \end{aligned} \quad (3.26)$$

Bukti

MSE dari penaksir UMVU $U\left(\theta^*\right)$ adalah

$$MSE\left[U\left(\theta^*\right)\right]$$

$$= \text{var}\left[U\left(\theta^*\right)\right] = E\left[U\left(\theta^*\right) - \{g(\theta)\}\right]^2$$

$$= E\left[\left(U\left(\theta^*\right) - U(\theta)\right) + \left(U(\theta) - \{g(\theta)\}\right)\right]^2$$

$$= E\left[\left(U\left(\theta^*\right) - U(\theta)\right)^2 + \left(U(\theta) - \{g(\theta)\}\right)^2 - 2\left[\left(U\left(\theta^*\right) - U(\theta)\right)\left(U(\theta) - \{g(\theta)\}\right)\right]\right] \quad (3.27)$$

Karena $U\left(\theta^*\right)$ adalah penaksir tak bias dari $g(\theta)$, maka persamaan (3.27) menjadi

$$MSE\left[U\left(\theta^*\right)\right] = E\left[U\left(\theta^*\right) - U(\theta)\right]^2 - \left[U(\theta) - \{g(\theta)\}\right]^2 \quad (3.28)$$

Dari (3.25) diperoleh

$$\left[U\left(\theta^*\right) - U(\theta)\right] = \frac{U'(\theta)\left(\theta^* - \theta\right)}{1} + \frac{U''(\theta)\left(\theta^* - \theta\right)^2}{2} + \frac{U'''(\theta)\left(\theta^* - \theta\right)^3}{6} + \frac{U''''(\theta)\left(\theta^* - \theta\right)^4}{24} + \cdots \quad (3.29)$$

Dengan mengambil Kuadrat pada kedua sisi persamaan (3.29) di peroleh

$$\left[U\left(\theta^*\right) - U(\theta)\right]^2 = \left[\frac{U'(\theta)\left(\theta^* - \theta\right)}{1} + \frac{U''(\theta)\left(\theta^* - \theta\right)^2}{2} + \frac{U'''(\theta)\left(\theta^* - \theta\right)^3}{6} + \frac{U''''(\theta)\left(\theta^* - \theta\right)^4}{24} + \cdots\right]^2$$

$$\left[U\left(\theta^*\right) - U(\theta)\right]^2$$

=

$$\left[\left[U'(\theta)^2\left(\theta-\theta^*\right)^2\right]+\left[U'(\theta)\frac{U''(\theta)}{2}\left(\theta-\theta^*\right)\left(\theta-\theta^*\right)^2\right]+\left[U'(\theta)\frac{U'''(\theta)}{6}\left(\theta-\theta^*\right)\left(\theta-\theta^*\right)^3\right]\right.$$

$$\left.+\left[U'(\theta)\frac{U''''(\theta)}{24}\left(\theta-\theta^*\right)\left(\theta-\theta^*\right)^4\right]+\ldots\right]$$

$$+\left[\left[U''(\theta)^2\left(\left(\theta-\theta^*\right)^2\right)^2\right]+\left[\frac{U''(\theta)}{2}U'(\theta)\left(\theta-\theta^*\right)^2\left(\theta-\theta^*\right)\right]\right.$$

$$\left.+\left[\frac{U''(\theta)}{2}\frac{U'''(\theta)}{6}\left(\theta-\theta^*\right)^2\left(\theta-\theta^*\right)^3\right]+\left[\frac{U''(\theta)}{2}\frac{U''''(\theta)}{24}\left(\theta-\theta^*\right)^2\left(\theta-\theta^*\right)^4\right]+\ldots\right]$$

$$+\left[\left[U'''(\theta)^2\left(\left(\theta-\theta^*\right)^3\right)^2\right]+\left[\frac{U'''(\theta)}{6}U'(\theta)\left(\theta-\theta^*\right)^3\left(\theta-\theta^*\right)\right]\right.$$

$$\left.+\left[\frac{U'''(\theta)}{6}\frac{U''(\theta)}{2}\left(\theta-\theta^*\right)^3\left(\theta-\theta^*\right)^2\right]+\left[\frac{U'''(\theta)}{6}\frac{U''''(\theta)}{24}\left(\theta-\theta^*\right)^3\left(\theta-\theta^*\right)^4\right]+\ldots\right]$$

$$+\left[\left[U''''(\theta)^2\left(\left(\theta-\theta^*\right)^4\right)^2\right]+\left[\frac{U''''(\theta)}{24}U'(\theta)\left(\theta-\theta^*\right)^4\left(\theta-\theta^*\right)\right]\right.$$

$$\left.+\left[\frac{U''''(\theta)}{24}\frac{U''(\theta)}{2}\left(\theta-\theta^*\right)^4\left(\theta-\theta^*\right)^2\right]+\left[\frac{U''''(\theta)}{24}\frac{U'''(\theta)}{6}\left(\theta-\theta^*\right)^4\left(\theta-\theta^*\right)^3\right]+\ldots\right] \quad (3.30)$$

Dengan mengambil ekspektasi pada kedua sisi persamaan (3.13) di peroleh

$$E\left[U\left(\theta^*\right)-U\left(\theta\right)\right]^2$$

$$=\left[\left[U'(\theta)^2 E\left[\left(\theta-\theta^*\right)^2\right]\right]+\left[U'(\theta)\frac{U''(\theta)}{2}E\left[\left(\theta-\theta^*\right)\left(\theta-\theta^*\right)^2\right]\right]+\left[U'(\theta)\frac{U'''(\theta)}{6}E\left[\left(\theta-\theta^*\right)\left(\theta-\theta^*\right)^3\right]\right]\right.$$

$$+\left[U'(\theta)\frac{U''''(\theta)}{24}E\left[\left(\theta-\theta^*\right)\left(\theta-\theta^*\right)^4\right]\right]+\ldots\Bigg]$$

$$+\left[\left[U''(\theta)^2 E\left(\left(\theta-\theta^*\right)^2\right)^2\right]+\left[\frac{U''(\theta)}{2}U'(\theta)E\left[\left(\theta-\theta^*\right)^2\left(\theta-\theta^*\right)\right]\right]\right.$$

$$+\left[\frac{U''(\theta)}{2}\frac{U'''(\theta)}{6}E\left[\left(\theta-\theta^*\right)^2\left(\theta-\theta^*\right)^3\right]\right]+\left[\frac{U''(\theta)}{2}\frac{U''''(\theta)}{24}E\left[\left(\theta-\theta^*\right)^2\left(\theta-\theta^*\right)^4\right]\right]+\ldots\Bigg]$$

$$+\left[\left[U'''(\theta)^2 E\left(\left(\theta-\theta^*\right)^3\right)^2\right]+\left[\frac{U'''(\theta)}{6}U'(\theta)E\left[\left(\theta-\theta^*\right)^3\left(\theta-\theta^*\right)\right]\right]\right.$$

$$+\left[\frac{U'''(\theta)}{6}\frac{U''(\theta)}{2}E\left[\left(\theta-\theta^*\right)^3\left(\theta-\theta^*\right)^2\right]\right]+\left[\frac{U'''(\theta)}{6}\frac{U''''(\theta)}{24}E\left[\left(\theta-\theta^*\right)^3\left(\theta-\theta^*\right)^4\right]\right]+\ldots\Bigg]$$

$$+\left[\left[U''''(\theta)^2 E\left(\left(\theta-\theta^*\right)^4\right)^2\right]+\left[\frac{U''''(\theta)}{24}U'(\theta)E\left[\left(\theta-\theta^*\right)^4\left(\theta-\theta^*\right)\right]\right]\right.$$

$$+\left[\frac{U''''(\theta)}{24}\frac{U''(\theta)}{2}E\left[\left(\theta-\theta^*\right)^4\left(\theta-\theta^*\right)^2\right]\right]+\left[\frac{U''''(\theta)}{24}\frac{U'''(\theta)}{6}E\left[\left(\theta-\theta^*\right)^4\left(\theta-\theta^*\right)^3\right]\right]+\ldots\Bigg]$$

Dengan menggunakan hasil sebelumnya, maka diperoleh

$$E\left[U\left(\theta^*\right)-U\left(\theta\right)\right]^2$$

$$=\left[\left[U'(\theta)^2\mu_2\right]+\left[U'(\theta)\frac{U''(\theta)}{2}\mu_3\right]+\left[U'(\theta)\frac{U'''(\theta)}{6}\mu_4\right]+\left[U'(\theta)\frac{U''''(\theta)}{24}E\left[\left(\theta-\theta^*\right)^5\right]\right]\right.$$

$$+\left[\frac{\left(U''(\theta)^2\right)}{4}\mu_4\right]+\left[\frac{U''(\theta)}{2}U'(\theta)\mu_3\right]+\left[\frac{U''(\theta)}{2}\frac{U'''(\theta)}{6}E\left[\left(\theta-\theta^*\right)^5\right]\right]$$

$$+\left[\frac{U''(\theta)}{2}\frac{U''''(\theta)}{24}E\left[\left(\theta-\theta^*\right)^6\right]\right]+\left[U'''(\theta)^2 E\left(\theta-\theta^*\right)^6\right]+\left[\frac{U'''(\theta)}{6}U'(\theta)\mu_4\right]$$

$$+\left[\frac{U'''(\theta)}{6}\frac{U''(\theta)}{2}E\left[\left(\theta-\theta^{*}\right)^{5}\right]\right]+\left[\frac{U'''(\theta)}{6}\frac{U''''(\theta)}{24}E\left[\left(\theta-\theta^{*}\right)^{7}\right]\right]$$

$$+\left[\left[U''''(\theta)^{2}E\left(\theta-\theta^{*}\right)^{8}\right]+\left[\frac{U''''(\theta)}{24}U'(\theta)E\left[\left(\theta-\theta^{*}\right)^{5}\right]\right]\right]$$

$$+\left[\frac{U''''(\theta)}{24}\frac{U''(\theta)}{2}E\left(\theta-\theta^{*}\right)^{6}\right]+\left[\frac{U''''(\theta)}{24}\frac{U'''(\theta)}{6}E\left(\theta-\theta^{*}\right)^{7}\right] \quad (3.31)$$

Sehingga,

$$E\left[U\left(\theta^{*}\right)-U(\theta)\right]^{2}$$

$$=\left[\left[U'(\theta)^{2}\mu_{2}\right]+\left[U'(\theta)\frac{U''(\theta)}{2}\mu_{3}\right]+\left[U'(\theta)\frac{U'''(\theta)}{6}\mu_{4}\right]\right.$$

$$\left.+\left[\frac{\left(U''(\theta)^{2}\right)}{4}\mu_{4}\right]+\left[\frac{U''(\theta)}{2}U'(\theta)\mu_{3}\right]+\left[\frac{U'''(\theta)}{6}U'(\theta)\mu_{4}\right]+O\left(n^{-3}\right)\right. \quad (3.32)$$

$$E\left[U\left(\theta^{*}\right)-U(\theta)\right]^{2}$$

$$=\left[U'(\theta)^{2}\mu_{2}\right]+\left[U'(\theta)U''(\theta)\right]\mu_{3}+\left[\frac{\{U''(\theta)\}^{2}}{4}+\frac{U'(\theta)U'''(\theta)}{3}\right]\mu_{4}+O\left(n^{-3}\right) \quad (3.33)$$

Karena $U\left(\theta^{*}\right)$ adalah penaksir tidak bias dari $g(\theta)$, maka dengan mengambil ekspektasi pada kedua sisi dari persamaan (3.29) di peroleh

$$\left[g(\theta)-U(\theta)\right]=\frac{U'(\theta)}{1}\mu_{1}+\frac{U''(\theta)}{2}\mu_{2}+\frac{U'''(\theta)\left(\theta^{*}-\theta\right)^{3}}{6}\mu_{3}+\frac{U''''(\theta)}{24}\mu_{4}+O\left(n^{-3}\right) \quad (3.34)$$

Dengan mensubtitusi hasil pada (3.7)-(3.10), persamaan (3.34) menjadi

$$\left[g(\theta)-U(\theta)\right]=\frac{U'(\theta)}{1}\left[\frac{b(\theta)}{n}\right]+\frac{U''(\theta)}{2}\left[\frac{1}{nI}+\frac{2\left[b'(\theta)\right]}{n^2 I}+\frac{\psi(\theta)}{n^2}+\frac{\left[b(\theta)\right]^2}{n^2}\right]$$

$$+\frac{U'''(\theta)}{6}\left[-\frac{(9K_{11}+7K_{30})}{2n^2 I^3}\right]+\frac{U''''(\theta)}{24}\left[\frac{3}{n^2 I^2}\right]+O\left(n^{-3}\right) \qquad (3.35)$$

Dengan menurunkan persamaan (3.35) dan mengabaikan order di atas $O\left(n^{-2}\right)$ diperoleh

$$\left[g'(\theta)-U'(\theta)\right]$$

$$=U''(\theta)\left[\frac{b(\theta)}{n}\right]+U'(\theta)\left[\frac{b'(\theta)}{n}\right]+\frac{U'''(\theta)}{2}\left[\frac{1}{nI}+\frac{2\left[b'(\theta)\right]}{n^2 I}+\frac{\psi(\theta)}{n^2}+\frac{\left[b(\theta)\right]^2}{n^2}\right]$$

$$+\frac{U''(\theta)}{2}\left[-\frac{(2K_{11}+K_{30})}{nI^2}+\frac{2b''(\theta)}{n^2 I}+\frac{2b'(\theta)\left[-(2K_{11}+K_{30})\right]}{n^2 I^2}+\frac{2b(\theta)}{n^2}+\frac{\psi'(\theta)}{n^2}\right]$$

$$+\frac{U''''}{6}\left[-\frac{(9K_{11}+7K_{30})}{2n^2 I^2}\right]+\frac{U'''''}{24}\left[\frac{3}{n^2 I^2}\right]+O\left(n^{-3}\right)$$

$$\left[g'(\theta)-U'(\theta)\right]$$

$$=U'(\theta)\left[\frac{b'(\theta)}{n}\right]+U''(\theta)\left[\frac{b(\theta)}{n}-\frac{2K_{11}+K_{30}}{2nI^2}\right]+U'''(\theta)\left[\frac{1}{2nI}\right]+O\left(n^{-2}\right) \qquad (3.36)$$

Sebagai catatan $(d/d\theta)[1/I]=-(2K_{11}+K_{30})/I^2$.

Karena, dari (3.28)

$$\frac{b(\theta)}{n}=-\frac{(K_{11}+K_{30})}{2nI^2}+O\left(n^{-2}\right)$$

Maka,

$$\left[g'(\theta)-U'(\theta)\right]$$

$$=U'(\theta)\left[\frac{b'(\theta)}{n}\right]+U''(\theta)\left[-\frac{(K_{11}+K_{30})}{2nI^2}-\frac{(2K_{11}+K_{30})}{2nI^2}\right]+U'''(\theta)\left[\frac{1}{2nI}\right]+O\left(n^{-2}\right)$$

$$=U'(\theta)\left[\frac{b'(\theta)}{n}\right]+U''(\theta)\left[-\frac{(3K_{11}+2K_{30})}{2nI^2}\right]+U'''(\theta)\left[\frac{1}{2nI}\right]+O\left(n^{-2}\right) \quad (3.37)$$

Sehingga,

$$U'(\theta)=g'(\theta)+U''(\theta)\left[\frac{(3K_{11}+2K_{30})}{2nI^2}\right]-U'''(\theta)\left[\frac{1}{nI}\right]-\frac{\{U'(\theta)\}\{b'(\theta)\}}{n}+O\left(n^{-2}\right) \quad (3.38)$$

dengan menurunkan persamaan (3.21), maka diperoleh

$$U''(\theta)\cong g''(\theta)+O\left(n^{-1}\right) \quad (3.39)$$

$$U'''(\theta)\cong g'''(\theta)+O\left(n^{-1}\right) \quad (3.40)$$

Kuadratkan (3.35), diperoleh

$$\left[g(\theta)-U(\theta)\right]^2=\left[\frac{U'(\theta)}{1}\left[\frac{b(\theta)}{n}\right]+\frac{U''(\theta)}{2}\left[\frac{1}{nI}+\frac{2\left[b'(\theta)\right]}{n^2I}+\frac{\psi(\theta)}{n^2}+\frac{\left[b(\theta)\right]^2}{n^2}\right]\right.$$

$$\left.+\frac{U'''(\theta)}{6}\left[-\frac{(9K_{11}+7K_{30})}{2n^2I^3}\right]+\frac{U''''(\theta)}{24}\left[\frac{3}{n^2I^2}\right]+O\left(n^{-3}\right)\right]^2$$

$$\left[g(\theta)-U(\theta)\right]^2$$

$$= U'(\theta)^2\left[\frac{b(\theta)}{n}\right]^2 + U'(\theta)\frac{U''(\theta)}{2}\left[\frac{b(\theta)}{n}\right]\left[\frac{1}{nI}+\frac{2\left[b'(\theta)\right]}{n^2I}+\frac{\psi(\theta)}{n^2}+\frac{\left[b(\theta)\right]^2}{n^2}\right]$$

$$+U'(\theta)\left[\frac{b(\theta)}{n}\right]\frac{U'''(\theta)}{6}\left[-\frac{(9K_{11}+7K_{30})}{2n^2I^3}\right]+U'(\theta)\frac{U''''(\theta)}{24}\left[\frac{b(\theta)}{n}\right]\left[\frac{3}{n^2I^2}\right]$$

$$+\left[\frac{U''(\theta)}{2}\right]^2\left[\frac{1}{nI}+\frac{2\left[b'(\theta)\right]}{n^2I}+\frac{\psi(\theta)}{n^2}+\frac{\left[b(\theta)\right]^2}{n^2}\right]^2+\frac{U''(\theta)}{2}U'(\theta)\left[\frac{b(\theta)}{n}\right]$$

$$\left[\frac{1}{nI}+\frac{2\left[b'(\theta)\right]}{n^2I}+\frac{\psi(\theta)}{n^2}+\frac{\left[b(\theta)\right]^2}{n^2}\right]+\frac{U''(\theta)}{2}\frac{U'''(\theta)}{6}\left[-\frac{(9K_{11}+7K_{30})}{2n^2I^3}\right]$$

$$\left[\frac{1}{nI}+\frac{2\left[b'(\theta)\right]}{n^2I}+\frac{\psi(\theta)}{n^2}+\frac{\left[b(\theta)\right]^2}{n^2}\right]+\frac{U''(\theta)}{2}\frac{U''''(\theta)}{24}\left[\frac{3}{n^2I^2}\right]$$

$$\left[\frac{1}{nI}+\frac{2\left[b'(\theta)\right]}{n^2I}+\frac{\psi(\theta)}{n^2}+\frac{\left[b(\theta)\right]^2}{n^2}\right]+\left[\frac{U'''(\theta)}{6}\right]^2\left[-\frac{(9K_{11}+7K_{30})}{2n^2I^3}\right]^2$$

$$+\frac{U'''(\theta)}{6}U'(\theta)\left[-\frac{(9K_{11}+7K_{30})}{2n^2I^3}\right]\left[\frac{b(\theta)}{n}\right]+\frac{U'''(\theta)}{6}\frac{U''(\theta)}{2}\left[-\frac{(9K_{11}+7K_{30})}{2n^2I^3}\right]$$

$$\left[g(\theta)-U(\theta)\right]^2$$

$$=\left\{g'(\theta)\right\}^2\left[\frac{(K_{11}+K_{30})^2}{4n^2I^4}\right]+\frac{\left\{g''(\theta)\right\}^2}{4n^2I^2}-\left(g'(\theta)\right)\left(g''(\theta)\right)\left[\frac{K_{11}+K_{30}}{2n^2I^3}\right]+O\left(n^{-3}\right) \quad (3.41)$$

Dengan mensubtitusi, hasil dari (3.38)-(3.40) dan hasil pada (3.7)-(3.10), maka persamaan (3.33) menjadi

$$E\left[U\left(\theta^*\right)-U(\theta)\right]^2 \quad =\left[\left\{g'(\theta)\right\}^2\right]\left[\frac{1}{nI}+\frac{2\left[b'(\theta)\right]}{n^2I}+\frac{\psi(\theta)}{n^2}+\frac{\left[b(\theta)\right]^2}{n^2}\right]$$

$$+\left[\frac{\{g'(\theta)\}\{g''(\theta)\}(3K_{11}+2K_{30})}{nI^2}-\frac{\{g'(\theta)\}\{g'''(\theta)\}}{nI}\right.$$

$$\left.-\frac{2\{g'(\theta)\}^2\left[b'(\theta)\right]}{n}\right]\left[\frac{1}{nI}\right]$$

$$+\left[\frac{\{g''(\theta)\}^2}{4}+\frac{\{g'(\theta)\}\{g'''(\theta)\}}{3}\right]\left[\frac{3}{n^2I^2}\right]$$

$$+\left[\{g'(\theta)\}\{g''(\theta)\}\right]\left[-\frac{(9K_{11}+7K_{30})}{2n^2I^2}\right]+O\left(n^{-3}\right)$$

$$=\{g'(\theta)\}^2\left[\frac{1}{nI}+\frac{\psi(\theta)}{n^2}+\frac{\left[b(\theta)\right]^2}{n^2}\right]+\left[\frac{\{g'(\theta)\}\{g''(\theta)\}}{2n^2I^3}\right.$$

$$\left.\left[-3\left(K_{11}+K_{30}\right)\right]\right]+\{g''(\theta)\}^2\left[\frac{3}{4n^2I^2}\right]+O\left(n^{-3}\right) \qquad (3.42)$$

Dengan menggunakan 3.41 dan 3.42, persamaan 3.28 menjadi,

$$MSE\left[U\left(\theta^*\right)\right]=\{g'(\theta)\}^2\left[\frac{1}{nI}+\frac{\psi(\theta)}{n^2}\right]-\{g'(\theta)\}\{g''(\theta)\}\frac{K_{11}+K_{30}}{n^2I^3}$$

$$+\left(g''(\theta)\right)^2\left[\frac{1}{2n^2I^2}\right]+O\left(n^{-3}\right) \qquad \blacksquare$$

Sehingga dapat disimpulkan bahwa

$$MSE\left[U\left(\theta^*\right)\right]=\{g'(\theta)\}^2\left[\frac{1}{nI}+\frac{\psi(\theta)}{n^2}\right]-\{g'(\theta)\}\{g''(\theta)\}\frac{K_{11}+K_{30}}{n^2I^3}$$

$$+\left(g''(\theta)\right)^2\left[\frac{1}{2n^2I^2}\right]+O\left(n^{-3}\right) \tag{3.43}$$

3.3 Perhitungan *Deficiency* dari Penaksir Maksimum Likelihood terhadap Penaksir UMVU

Setelah diperoleh hasil $MSE\left[g\left(\theta^*\right)\right]$ dan $MSE\left[U\left(\theta^*\right)\right]$ maka dapat dicari nilai dari *deficiency*. Berikut akan ditunjukan nilai *deficiency* dari penaksir maksimum Likelihood terhadap penaksir UMVU

Teorema 3

Deficiency dari penaksir maksimum Likelihood $g\left(\theta^*\right)$ terhadap penaksir UMVU $U\left(\theta^*\right)$ ditunjukan sebagai berikut :

$$D_{HL}\left[g\left(\theta^*\right),U\left(\theta^*\right)\right]$$

$$=-\left[\frac{(7K_{11}+5K_{30})}{2I^2}\right]\frac{\left\{g''(\theta)\right\}}{\left\{g'(\theta)\right\}}+\frac{1}{I}\left[\frac{\left\{g'''(\theta)\right\}}{\left\{g'(\theta)\right\}}+\frac{1}{4}\left\{\frac{g''(\theta)}{g'(\theta)}\right\}^2\right]+\left[2b'(\theta)+I\left[b(\theta)\right]^2\right] \tag{3.44}$$

Bukti :

Dari persamaan (3.24) dan (3.43) telah diketahui bahwa

$$MSE\left\{g\left(\theta^*\right)\right\}$$

$$=\left\{g'(\theta)\right\}^2\left[\frac{1}{nI}+\frac{\psi(\theta)}{n^2}+\frac{2b'(\theta)}{n^2I}+\frac{\left[b(\theta)\right]^2}{n^2}\right]-\left\{g'(\theta)\right\}\left\{g''(\theta)\right\}\left[\frac{(9K_{11}+7K_{30})}{2n^2I^3}\right]$$

$$+\left\{g''(\theta)\right\}\left[\frac{3}{4n^2I^2}\right]+\left\{g'(\theta)\right\}\left\{g'''(\theta)\right\}\left[\frac{1}{n^2I^2}\right]+O\left(n^{-3}\right) \qquad (3.24)$$

Dan,

$$MSE\left\{U\left(\theta^*\right)\right\}$$

$$=\left\{g'(\theta)\right\}^2\left[\frac{1}{nI}+\frac{\psi(\theta)}{n^2}\right]+\left\{g'(\theta)\right\}\left\{g''(\theta)\right\}\left[-\frac{(K_{11}+K_{30})}{n^2I^3}\right]+\left\{g''(\theta)\right\}^2\left[\frac{1}{2n^2I^2}\right]$$

$$+O\left(n^{-3}\right) \qquad (3.43)$$

Dengan asumsi bahwa MLE dan UMVU identik, maka dapat dicari *deficiency* dari penaksir Maksimum likelihood terhadap penaksir UMVU dengan cara menyamakan kedua buah persamaan di atas sebagai berikut :

$$D_{HL}\left[g\left(\theta^*\right),U\left(\theta^*\right)\right]=$$

MSE $g\left(\theta^*\right)$ = MSE $U\left(\theta^*\right)$

$$\Leftrightarrow \quad \left\{g'(\theta)\right\}^2\left[\frac{1}{nI}+\frac{\psi(\theta)}{n^2}+\frac{2b'(\theta)}{n^2I}+\frac{\left[b(\theta)\right]^2}{n^2}\right]-\left\{g'(\theta)\right\}\left\{g''(\theta)\right\}\left[\frac{(9K_{11}+7K_{30})}{2n^2I^3}\right]$$

$$+\left\{g''(\theta)\right\}\left[\frac{3}{4n^2I^2}\right]+\left\{g'(\theta)\right\}\left\{g'''(\theta)\right\}\left[\frac{1}{n^2I^2}\right]+O\left(n^{-3}\right)=\left\{g'(\theta)\right\}^2\left[\frac{1}{nI}+\frac{\psi(\theta)}{n^2}\right]$$

$$+\left\{g'(\theta)\right\}\left\{g''(\theta)\right\}\left[-\frac{(K_{11}+K_{30})}{n^2I^3}\right]+\left\{g''(\theta)\right\}^2\left[\frac{1}{2n^2I^2}\right]+O\left(n^{-3}\right)$$

$$\Leftrightarrow \quad \left\{g^{'}(\theta)\right\}^{2}\left[\frac{2b^{'}(\theta)}{n^{2}I}+\frac{\left[b(\theta)\right]^{2}}{n^{2}}\right]+\left\{g^{'}(\theta)\right\}\left\{g^{''}(\theta)\right\}\left[-\frac{(7K_{11}+5K_{30})}{2n^{2}I^{3}}\right]$$

$$\left\{g^{''}(\theta)\right\}^{2}\left[\frac{1}{4n^{2}I^{2}}\right]+\left\{g^{'}(\theta)\right\}\left\{g^{'''}(\theta)\right\}\left[\frac{1}{n^{2}I^{2}}\right]+O\left(n^{-3}\right) \tag{3.45}$$

Kalikan persamaan (3.45) dengan $n^{2}I$

$$\Leftrightarrow \quad \left\{g^{'}(\theta)\right\}^{2}\left[2b^{'}(\theta)+I\left[b(\theta)\right]^{2}\right]+\left\{g^{'}(\theta)\right\}\left\{g^{''}(\theta)\right\}\left[-\frac{(7K_{11}+5K_{30})}{2I^{2}}\right]$$

$$+\left\{g^{''}(\theta)\right\}^{2}\left[\frac{1}{4I}\right]+\left\{g^{'}(\theta)\right\}\left\{g^{'''}(\theta)\right\}\left[\frac{1}{I}\right]+O\left(n^{-3}\right) \tag{3.46}$$

Kalikan persamaan di atas dengan $\dfrac{1}{\left\{g^{'}(\theta)\right\}^{2}}$

$$\Leftrightarrow \quad \left[2b^{'}(\theta)+I\left[b(\theta)\right]^{2}\right]+\frac{\left\{g^{''}(\theta)\right\}}{\left\{g^{'}(\theta)\right\}}\left[-\frac{(7K_{11}+5K_{30})}{2I^{2}}\right]$$

$$+\left\{\frac{g^{''}(\theta)}{g^{'}(\theta)}\right\}^{2}\left[\frac{1}{4I}\right]+\frac{\left\{g^{'''}(\theta)\right\}}{\left\{g^{'}(\theta)\right\}}\left[\frac{1}{I}\right]$$

$$\Leftrightarrow \quad -\left[\frac{(7K_{11}+5K_{30})}{2I^{2}}\right]\frac{\left\{g^{''}(\theta)\right\}}{\left\{g^{'}(\theta)\right\}}+\frac{1}{I}\left[\frac{\left\{g^{'''}(\theta)\right\}}{\left\{g^{'}(\theta)\right\}}+\frac{1}{4}\left\{\frac{g^{''}(\theta)}{g^{'}(\theta)}\right\}^{2}\right]+\left[2b^{'}(\theta)+I\left[b(\theta)\right]^{2}\right]$$

■

Sehingga dapat disimpulkan, bahwa

$$D_{HL}\left[g\left(\theta^{*}\right),U\left(\theta^{*}\right)\right]=-\left[\frac{(7K_{11}+5K_{30})}{2I^{2}}\right]\frac{\left\{g^{''}(\theta)\right\}}{\left\{g^{'}(\theta)\right\}}+\frac{1}{I}\left[\frac{\left\{g^{'''}(\theta)\right\}}{\left\{g^{'}(\theta)\right\}}+\frac{1}{4}\left\{\frac{g^{''}(\theta)}{g^{'}(\theta)}\right\}^{2}\right]$$

$$+\left[2b'(\theta)+I\left[b(\theta)\right]^2\right] \tag{3.47}$$

Jika,

$$g_1(\theta) = \left\{\frac{g''(\theta)}{g'(\theta)}\right\}$$

$$g_2(\theta) = \left\{\frac{g'''(\theta)}{g'(\theta)}+\frac{1}{4}\left\{\frac{g''(\theta)}{g'(\theta)}\right\}^2\right\} = \left\{\frac{g'''(\theta)}{g'(\theta)}+\frac{1}{4}\left[g_1(\theta)\right]\right\}$$

Dan

$$g_3(\theta) = \left[2b'(\theta)+I\left[b(\theta)\right]^2\right]$$

Maka , persamaan (3.47) dapat ditulis sebagai berikut

$$D_{HL}\left[g\left(\theta^*\right),U\left(\theta^*\right)\right] = -\left[\frac{(7K_{11}+5K_{30})}{2I^2}\right]g_1(\theta)+\frac{g_2(\theta)}{I}+g_3(\theta) \tag{3.48}$$

Pada persamaan tersebut $g_3(\theta)$ menunjukan bias pada penaksir maksimum Likelihood. Dan jika ternyata penaksir maksimum Likelihood tidak bias maka $g_3(\theta)$ bernilai nol.

3.4 *Deficiency* dari Penaksir Maksimum Likelihood terhadap penaksir UMVU pada persamaaan $g(\theta),\theta$, dan turunannya.

Pada bagian ini,, akan ditunjukan nilai *deficiency* dari penaksir maksimum Likelihood terhadap penaksir UMVU pada persamaan $g(\theta),\theta$, dan turunannya.

Karena,

$$\int \exp\left(\phi_1(\theta)T(x)+\phi_2(\theta)+Q(x)\right)dx=1 \tag{3.49}$$

Dengan asumsi,

$\eta(\theta)\phi_1'(\theta)+\phi_2'(\theta)=0$ dan $\phi_1'(\theta)>0$, untuk setiap $\theta\in\Theta$

Maka,

$$\left[-\frac{\phi_2'(\theta)}{\phi_1'(\theta)}\right]=\eta(\theta) \tag{3.50}$$

Sehingga, dengan menggunakan hasil pada (Gudi dan Nagnur, 2004), diperoleh

$$\begin{aligned} E\left[T(X)\right] &= \int\{T(x)\}\exp\left(\phi_1(\theta)T(x)+\phi_2(\theta)+Q(x)\right)dx \\ &= \{\eta(\theta)\} \end{aligned} \tag{3.51}$$

$$\begin{aligned} E\left[T(X)\right]^2 &= \int\{T(x)\}^2\exp\left(\phi_1(\theta)T(x)+\phi_2(\theta)+Q(x)\right)dx \\ &= \{\eta(\theta)\}^2+\frac{\{\eta'(\theta)\}}{\phi_1'(\theta)} \end{aligned} \tag{3.52}$$

$$E\left[T(X)\right]^3 = \int\{T(x)\}^3\exp\left(\phi_1(\theta)T(x)+\phi_2(\theta)+Q(x)\right)dx$$

$$=\{\eta(\theta)\}^3+\frac{3\{\eta(\theta)\}\{\eta'(\theta)\}}{[\phi_1'(\theta)]}-\frac{\{\eta'(\theta)\}[\phi_1''(\theta)]}{[\phi_1'(\theta)]^3}+\frac{\{\eta''(\theta)\}}{[\phi_1'(\theta)]^2} \quad (3.53)$$

dan

$$I=E\left[-\frac{\partial^2}{\partial\theta^2}[\log f(X;\theta)]\right]=\{\phi_1'(\theta)\}\{\eta'(\theta)\} \quad (3.54)$$

$$K_{ij}=\{\phi_1'(\theta)\}^i\{\phi_1''(\theta)\}^j E[T(X)-\eta(\theta)]^{i+j},\forall i,j. \quad (3.55)$$

Dengan menggunakan (3.51) dan (3.52) diperoleh nilai dari $\mathrm{var}[T(x)]$ yaitu,

$$\mathrm{var}[T(x)]=\frac{\{\eta'(\theta)\}}{[\phi_1'(\theta)]} \quad (3.58)$$

Bukti

$$\mathrm{var}[T(x)]=E\left(T(x)^2\right)-\left(E(T(X))\right)^2$$

$$=\left[\{\eta(\theta)\}^2+\frac{\{\eta'(\theta)\}}{\phi_1'(\theta)}\right]-\{\eta(\theta)\}^2$$

$$=\frac{\{\eta'(\theta)\}}{[\phi_1'(\theta)]}$$

■

Dari (3.55) diperoleh

$$K_{11}=\{\phi_1'(\theta)\}^1\{\phi_1''(\theta)\}^1 E[T(X)-\eta(\theta)]^2$$

$$=\{\phi_1'(\theta)\}\{\phi_1''(\theta)\}\left[E[T(X)]^2-2E[T(X)]\eta(\theta)+\{\eta(\theta)\}^2\right]$$

$$= \{\phi_1'(\theta)\}\{\phi_1''(\theta)\}\left[\{\eta(\theta)\}^2 + \frac{\eta'(\theta)}{\phi_1'(\theta)} - 2\{\eta(\theta)\}^2 + \{\eta(\theta)\}^2\right]$$

$$= \left[\phi_1''(\theta)\right]\{\eta'(\theta)\} \quad (3.59)$$

$$K_{02} = \{\phi_1'(\theta)\}^0 \{\phi_1''(\theta)\}^2 E\left[T(X) - \eta(\theta)\right]^2$$

$$= \{\phi_1''(\theta)\}^2 \left[E\left[T(X)\right]^2 - 2E\left[T(X)\right]\eta(\theta) + \{\eta(\theta)\}^2\right]$$

$$= \{\phi_1''(\theta)\}^2 \left[\{\eta(\theta)\}^2 + \frac{\eta'(\theta)}{\phi_1'(\theta)} - 2\{\eta(\theta)\}^2 + \{\eta(\theta)\}^2\right]$$

$$= \left[\phi_1''(\theta)\right]\frac{\{\eta'(\theta)\}}{\left[\phi_1'(\theta)\right]} \quad (3.60)$$

dan

$$K_{30} = \{\phi_1'(\theta)\}^3 \{\phi_1''(\theta)\}^0 E\left[T(X) - \eta(\theta)\right]^3$$

$$= \{\phi_1'(\theta)\}^3 \left[E\left[T(X)\right]^3 - 3E\left[T(X)\right]^2 \eta(\theta) + 3E\left[T(X)\right]\eta(\theta)^2 - \{\eta(\theta)\}^3\right]$$

$$= \{\phi_1'(\theta)\}^3 \left[\left[\{\eta(\theta)\}^3 + \frac{3\{\eta(\theta)\}\{\eta'(\theta)\}}{\left[\phi_1'(\theta)\right]} - \frac{\{\eta'(\theta)\}\left[\phi_1''(\theta)\right]}{\left[\phi_1'(\theta)\right]^3} + \frac{\{\eta''(\theta)\}}{\left[\phi_1'(\theta)\right]^2}\right.\right.$$

$$\left. -3\eta(\theta)^2\eta(\theta) - \frac{3\eta(\theta)\eta'(\theta)}{\phi_1'(\theta)} + 3\eta(\theta)\eta(\theta)^2 - \eta(\theta)^3\right]$$

$$= \left[-\phi_1''(\theta)\right]\{\eta'(\theta)\} + \left[\phi_1'(\theta)\right]\{\eta''(\theta)\}$$

$$= -K_{11} + \left[\phi_1'(\theta)\right]\{\eta''(\theta)\} \quad (3.61)$$

Dengan menggunakan 3.54,3.59-3.61, beberapa hasil berikut diperoleh

$$b(\theta) = -\frac{(K_{11}+K_{30})}{2I^2}$$

$$= -\frac{\phi'(\theta)\eta''(\theta)}{2\left[\phi_1'(\theta)\eta'(\theta)\right]^2}$$

$$= -\frac{\eta''(\theta)}{2\left[\phi_1'(\theta)\right]\left\{\eta'(\theta)\right\}^2} \tag{3.62}$$

Dan

$$\psi(\theta) = \frac{2\left(IK_{02}-K_{11}{}^2\right)+\left(K_{11}+K_{30}\right)^2}{2I^4}$$

$$= \frac{\left[2\left[\phi_1'(\theta)\eta'(\theta)\right]\left[\phi_1''(\theta)\frac{\eta'(\theta)}{\phi_1'(\theta)}\right]-\left[\phi_1''(\theta)\eta'(\theta)\right]\right]+\eta'(\theta)}{2\left[\phi_1'(\theta)\right]^2\left\{\eta'(\theta)\right\}^4}$$

$$= \frac{\eta''(\theta)}{2\left[\phi_1'(\theta)\right]^2\left\{\eta'(\theta)\right\}^4} \tag{3.63}$$

Dengan menurunkan 3.26 terhadap θ, diperoleh

$$b'(\theta) = -\left[\frac{\eta'''(\theta)2\phi_1'(\theta)\left\{\eta'(\theta)\right\}^2-\eta''(\theta)\left(2\phi_1''(\theta)\left\{\eta'(\theta)\right\}^2+4\phi_1'(\theta)\eta''(\theta)\eta'(\theta)\right)}{\left[2\phi_1'(\theta)\left\{\eta'(\theta)\right\}^2\right]^2}\right]$$

$$= \left[\frac{-\left(\eta'''(\theta)2\phi_1'(\theta)\left\{\eta'(\theta)\right\}^2\right)+2\phi_1''(\theta)\left\{\eta'(\theta)\right\}^2\eta''(\theta)+4\phi_1'(\theta)\eta''(\theta)^2\eta'(\theta)}{\left[4\phi_1'(\theta)^2\left\{\eta'(\theta)\right\}^4\right]}\right]$$

$$= -\frac{\{\eta'''(\theta)\}}{[\phi_1'(\theta)]\{\eta'(\theta)\}^2} + \frac{\{\eta''(\theta)\}[\phi_1''(\theta)]}{[\phi_1'(\theta)]^2\{\eta'(\theta)\}^2} + \frac{\{\eta''(\theta)\}^2}{4[\phi_1'(\theta)]\{\eta'(\theta)\}^3} \qquad (3.64)$$

dengan mensubtitusikan hasil-hasil di atas pada persamaan 3.24 dan 3.43 diperoleh

$$MSE\left[g\left(\theta^*\right)\right]$$

$$= \{g'(\theta)\}^2\left[\frac{1}{nI} + \frac{2b'(\theta)}{n^2 I} + \frac{\psi(\theta)}{n^2} + \frac{(K_{11}+K_{30})^2}{4n^2 I^2}\right] - \left[\left(g'(\theta)g''(\theta)\right)\right]$$

$$\left[\frac{4K_{11}+3K_{30}}{n^2 I^3} + \frac{K_{11}+K_{30}}{2n^2 I^3}\right] + \left(g''(\theta)\right)^2\left[\frac{1}{2n^2 I^2} + \frac{1}{4n^2 I^2}\right] + g'(\theta)g'''(\theta)\left[\frac{1}{n^2 I^2}\right]$$

$$= \{g'(\theta)\}^2\left[\frac{1}{n}\left\{\frac{1}{\phi_1'(\theta)\{\eta'(\theta)\}}\right\} + \frac{1}{n^2}\left\{-\frac{\eta'''(\theta)}{[\phi_1'(\theta)]^2\{\eta'(\theta)\}^3}\right.\right.$$

$$\left.\left. + \frac{[\phi_1''(\theta)]\{\eta''(\theta)\}}{[\phi_1'(\theta)]^3\{\eta'(\theta)\}^3} + \frac{11}{4}\frac{\{\eta''(\theta)\}^2}{[\phi_1'(\theta)]^2\{\eta'(\theta)\}^4}\right\}\right]$$

$$+ \frac{\{g'(\theta)\}\{g''(\theta)\}}{n^2}\left\{-\frac{[\phi_1''(\theta)]}{[\phi_1'(\theta)]^3\{\eta'(\theta)\}^2} - \frac{7}{2}\frac{\{\eta''(\theta)\}}{[\phi_1'(\theta)]^2\{\eta'(\theta)\}^3}\right\}$$

$$+ \frac{g''(\theta)^2}{n^2}\left\{\frac{3}{4[\phi_1'(\theta)]^2\{\eta'(\theta)\}^2}\right\} + \frac{\{g'(\theta)\}\{g'''(\theta)\}}{n^2}\left\{\frac{1}{[\phi_1'(\theta)]^2\{\eta'(\theta)\}^2}\right\}$$

$$+ O\left(n^{-3}\right) \qquad (3.65)$$

Dan

$$MSE\left[U\left(\theta^*\right)\right]$$

$$=\{g'(\theta)\}^2\left[\frac{1}{nI}+\frac{\psi(\theta)}{n^2}\right]+2\{g'(\theta)\}\{g''(\theta)\}\left[-\frac{(K_{11}+K_{30})}{n^2I^3}\right]$$

$$+\{g''(\theta)\}^2\left[\frac{1}{2n^2I^2}\right]+O\left(n^{-3}\right)$$

$$=\{g'(\theta)\}^2\left[\frac{1}{n}\left\{\frac{1}{\left[\phi_1'(\theta)\right]\{\eta'(\theta)\}}\right\}+\frac{1}{n^2}\left\{\frac{\left\{\eta''(\theta)^2\right\}}{2\left[\phi_1'(\theta)\right]^2\{\eta'(\theta)\}^4}\right\}\right]$$

$$+\frac{g'(\theta)\,g''(\theta)}{n^2}\left\{-\frac{\eta''(\theta)}{\left[\phi_1'(\theta)\right]^2\{\eta'(\theta)\}^3}\right\}$$

$$+\frac{\{g''(\theta)\}^2}{n^2}\left\{\frac{1}{2\left[\phi_1'(\theta)\right]^2\{\eta'(\theta)\}^2}\right\}+O\left(n^{-3}\right) \qquad (3.66)$$

Dan akhirnya,

$$D_{HL}\left[g\left(\theta^*\right),U\left(\theta^*\right)\right]=$$

$$\text{MSE } g\left(\theta^*\right)=\text{MSE } U\left(\theta^*\right)$$

$$\Leftrightarrow \quad \{g'(\theta)\}^2\left[\frac{1}{n}\left\{\frac{1}{\phi_1'(\theta)\{\eta'(\theta)\}}\right\}+\frac{1}{n^2}\left\{-\frac{\eta'''(\theta)}{\left[\phi_1'(\theta)\right]^2\{\eta'(\theta)\}^3}\right.\right.$$

$$\left.\left.+\frac{\left[\phi_1''(\theta)\right]\{\eta''(\theta)\}}{\left[\phi_1'(\theta)\right]^3\{\eta'(\theta)\}^3}+\frac{11}{4}\frac{\{\eta''(\theta)\}^2}{\left[\phi_1'(\theta)\right]^2\{\eta'(\theta)\}^4}\right\}\right]$$

$$+\frac{\{g'(\theta)\}\{g''(\theta)\}}{n^2}\left\{-\frac{[\phi_1''(\theta)]}{[\phi_1'(\theta)]^3\{\eta'(\theta)\}^2}-\frac{7}{2}\frac{\{\eta''(\theta)\}}{[\phi_1'(\theta)]^2\{\eta'(\theta)\}^3}\right\}$$

$$+\frac{g''(\theta)^2}{n^2}\left\{\frac{3}{4[\phi_1'(\theta)]^2\{\eta'(\theta)\}^2}\right\}+\frac{\{g'(\theta)\}\{g'''(\theta)\}}{n^2}\left\{\frac{1}{[\phi_1'(\theta)]^2\{\eta'(\theta)\}^2}\right\}$$

$$=\{g'(\theta)\}^2\left[\frac{1}{n}\left\{\frac{1}{[\phi_1'(\theta)]\{\eta'(\theta)\}}\right\}+\frac{1}{n^2}\left\{\frac{\{\eta''(\theta)^2\}}{2[\phi_1'(\theta)]^2\{\eta'(\theta)\}^4}\right\}\right]$$

$$+\frac{g'(\theta)g''(\theta)}{n^2}\left\{-\frac{\eta''(\theta)}{[\phi_1'(\theta)]^2\{\eta'(\theta)\}^3}\right\}$$

$$+\frac{\{g''(\theta)\}^2}{n^2}\left\{\frac{1}{2[\phi_1'(\theta)]^2\{\eta'(\theta)\}^2}\right\}+O\left(n^{-3}\right)$$

$$\Leftrightarrow\quad g_1(\theta)\left[-\frac{\phi''(\theta)}{[\phi_1'(\theta)]^2\{\eta'(\theta)\}}-\frac{5}{2}\frac{\{\eta''(\theta)\}}{[\phi_1'(\theta)]\{\eta'(\theta)\}^2}\right]$$

$$+g_2(\theta)\left[\frac{1}{[\phi_1'(\theta)]\{\eta'(\theta)\}}\right]+g_3(\theta)$$

Sehingga dapat disimpulkan bahwa *deficiency* dari Penaksir Maksimum Likelihood terhadap penaksir UMVU pada persamaaan $g(\theta),\theta,$ dan turunannya adalah:

$$D_{HL}\left[g\left(\theta^*\right),U\left(\theta^*\right)\right]=g_1(\theta)\left[-\frac{\phi'''(\theta)}{\left[\phi_1'(\theta)\right]^2\left\{\eta'(\theta)\right\}}-\frac{5}{2}\frac{\left\{\eta''(\theta)\right\}}{\left[\phi_1'(\theta)\right]\left\{\eta'(\theta)\right\}^2}\right]$$

$$+g_2(\theta)\left[\frac{1}{\left[\phi_1'(\theta)\right]\left\{\eta'(\theta)\right\}}\right]+g_3(\theta) \qquad (3.67)$$

Jika diperhatikan nilai *deficiency* pada persamaan (3.67) bergantung pada nilai $g(\theta),\phi_1(\theta),\phi_2(\theta),\eta(\theta)$ dan turunannya terhadap θ